Subdivisions, shellability, and the Zeeman conjecture

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Abstract

We prove the $(d-2)$-nd barycentric subdivision of every convex d-complex is shellable. This yields a new characterization of the PL property in terms of shellability: A triangulation of a sphere or of a ball is PL if and only if it becomes shellable after sufficiently many barycentric subdivisions. This improves results by Whitehead, Zeeman and Glaser.

We also show that any contractible complex can be made collapsible by repeatedly taking products with an interval. This strengthens results by Dierker and Lickorish.

Finally, we show that the Zeeman conjecture is equivalent to the statement "the product of any contractible 2-complex with an interval becomes (simplicially) collapsible after a suitable number of barycentric subdivisions". This number cannot be bounded: For any two positive integers m, n, we construct a 2-complex C such that $\mathrm{sd}^{\,m}(C \times I^n)$ is not collapsible.

1 Introduction

Shellability is one of the earliest notion in combinatorial topology: For a brief history, see Ziegler [31, 32]. Convexity plays a crucial role in most mathematical fields, cf. e.g. Barvinok [4]. There is a well-known connection between the two notions: The boundary of every convex polytope is a shellable sphere. This fact was already known in the 1850's and widely used in the 1880's [18, p. 141], although apparently no formal proof appeared until 1970 [7]. One can hope to establish the shellability property for convex complexes themselves, and not just for their boundaries. Unfortunately, the hope was lost already in 1958, when Mary E. Rudin described an unshellable subdivision of a tetrahedron [26].

In this paper, we resuscitate this hope. The new idea is to allow a limited number of barycentric subdivisions.

Main Theorem 1 (Corollary 4.18). *Let C be any convex d-dimensional polytopal complex. The $(d-2)$-nd barycentric subdivision of C is shellable.*

The bound is best possible, at least in low dimensions: The $(d-3)$-rd barycentric subdivision of a convex d-complex need not be shellable, as shown for $d = 3$ by Rudin's ball.

Barycentric subdivisions preserve plenty of combinatorial properties, including collapsibility and shellability. Main Theorem 1, which improves a result in [1], shows that they even induce these properties, after a relatively small number of steps. In contrast, triangulations of I^d can be

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arbitrarily nasty when $d \geq 3$. For example, for each m there is a 3-ball whose m-th barycentric subdivision is still unshellable [17].

An easy consequence of Main Theorem 1 is the following:

Main Theorem 2 (Theorems 3.4 & 5.1). *A simplicial complex has some shellable subdivision if and only if it becomes shellable after suitably many barycentric subdivisions. The same holds if we replace "shellable" with "collapsible".*

Main Theorem 2 allows us to simplify the classic definitions of PL topology. In the Forties, Whitehead proved that every PL ball admits a collapsible subdivision. With a more delicate construction, Glaser showed in 1970 that every PL ball admits also a collapsible stellar subdivision [16, pp. 58–69]. In 1987, Pachner characterized PL d-balls as those obtainable from the d-simplex via a finite sequence of moves, called "bistellar flips" or later also "Pachner moves" [24]. This opened doors to algorithmic approaches, see e.g. Mijatović [23] or Burton [8].

Using Main Theorem 2, we can now fully characterize the PL property via shellability of iterated barycentric subdivisions, thus answering a question by Ed Swartz and Lou Billera to the second author; cf. [5].

Main Theorem 3 (Theorem 5.2). *A triangulated ball or sphere is PL if and only if it becomes shellable after a suitable number of barycentric subdivisions. This number can be arbitrarily high.*

Similarly, one can characterize the PL notion for manifolds; see Theorem 5.4. Note that, for fixed $d \geq 3$, it is unknown whether triangulations of PL d-balls with N facets are exponentially many (in N) or more. Since the triangulations with shellable i-th barycentric subdivision are exponentially many, one may regard the class of PL d-balls as limit of an increasing sequence of subclasses of exponential size (Proposition 5.3).

Our methods apply also to the Zeeman conjecture, which claims that the product of any contractible 2-dimensional complex C with the interval I has a collapsible subdivision. Our first result is that any contractible complex becomes collapsible after taking the product with a cube of sufficiently high dimension:

Main Theorem 4 (Cor. 3.22 & 3.15). *For any contractible complex C, there is an integer n such that $C \times I^n$ is polytopally collapsible. This n depends on C, and can be arbitrarily large.*

This was already conjectured by Bob Oliver in a 1998 email to Anders Björner. (We thank Anders Björner for this communication.) Main Theorem 4 can be viewed as a strengthening of analogous results in the PL category by Dierker [12] and Lickorish [20, Lemma 1]. We also give a concrete example of a complex that becomes non-evasive when multiplied with an interval, thereby solving a question by Welker [27, Open Problem 2]: See Proposition 3.24.

The Zeeman conjecture has been open since the Sixties, although some special cases have been solved [11, 15]. Its importance is highlighted by the fact that it implies, quite straightforwardly [16, pp. 78–79], the three-dimensional Poincaré conjecture, recently proven by Perelman. In turn, Perelman's work shows the validity of the Zeeman conjecture for a large class of contractible 2-complexes [22].

Via Main Theorem 2, we can now rephrase the Zeeman conjecture in a rather concrete way:

Main Theorem 5 (Prop. 3.7 & Cor. 3.15). *The Zeeman conjecture is equivalent to the claim that for any contractible 2-complex C there is an integer m such that $\operatorname{sd}^m(C \times I)$ is collapsible. However, for any $m, n \in \mathbb{N}$, we can concretely describe a contractible 2-complex $C_{m,n}$ such that the m-th barycentric subdivision of $C_{m,n} \times I^n$ is* not *collapsible.*

2 Notation

A *polytope* (in $\mathbb{R}^d$) is the convex hull of finitely many points in $\mathbb{R}^d$. Similarly, if S^d is the unit sphere in $\mathbb{R}^{d+1}$, a *polytope in* S^d is the convex hull of a finite number of points in some open hemisphere of S^d. Polytopes in S^d are in one-to-one correspondence with polytopes in $\mathbb{R}^d$. [1]

A *(polytopal) complex* in $\mathbb{R}^d$ (resp. in S^d) is a finite nonempty collection of polytopes in $\mathbb{R}^d$ (resp. S^d), such that the intersection of any two polytopes is a face of both. The polytopes in a complex are called *faces*; the inclusion-maximal faces are called *facets*. A polytopal complex is *pure* if all its facets are of the same dimension. The *face poset* of a complex is the partially ordered set formed by its faces, with respect to inclusion. Two polytopal complexes are called *combinatorially equivalent* if their face posets are isomorphic as posets. The *underlying space* $|C|$ of a polytopal complex C is the union of its faces as topological space. If two complexes are combinatorially equivalent, their underlying spaces are homeomorphic.

Let C be a polytopal complex in $\mathbb{R}^d$, and let A be a subset of $\mathbb{R}^d$. We denote by $\mathcal{R}(C, A)$ the "restriction of C to A", namely, the inclusion-maximal subcomplex D of C such that $|D| \subset A$, where $|D|$ is the natural embedding of D inside the given realization of C. The notation $\mathcal{R}(C, A)$ makes sense only if the embedding of C is specified. Therefore, throughout the present paper, we shall always consider polytopal complexes together with a **fixed** geometric realization in $\mathbb{R}^d$ (or S^d), for a specific d.

If C is a polytopal complex, a *subdivision* of C is any polytopal complex C' with $|C| = |C'|$, such that the underlying space of each face of C' is contained in the underlying space of some face of C. Subdivisions always exist. A *simplicial complex* is a polytopal complex all whose faces are simplices. A *triangulation* of a topological space X (resp. of a polytopal complex P) is any simplicial complex whose underlying space is homeomorphic to X (resp. to $|P|$). Every simplicial subdivision is a triangulation, but the converse is false: If we remove the interior of some 5-simplex Δ from Edwards' non-PL triangulation of S^5 [13], the resulting complex is a triangulation of Δ, but not a subdivision of Δ.

A *derived subdivision* of a polytopal complex P is any subdivision of P with face poset isomorphic to the order complex of the face poset of P. Any derived subdivision is both a subdivision and a triangulation. An example of a derived subdivision is the *barycentric subdivision* $\mathrm{sd}\, P$, which uses as vertices the barycenters of all faces of P. (Other choices are possible, e.g. the "E-splitting subdivision" which will be introduced in Definition 4.15.) Barycentric subdivisions can be iterated, by setting $\mathrm{sd}^{\,0} P := P$ and recursively $\mathrm{sd}^{\,m+1}(P) := \mathrm{sd}\,(\mathrm{sd}^{\,m} P)$. As m grows the mesh gets finer and finer: For any subdivision K' of K, there is a natural number m such that the m-th barycentric subdivision of K is also a subdivision of K' [30, Lemma 4].

In case P is a subcomplex of a polytopal complex Q, the *m-th derived neighborhood of P in Q*, denoted by $N^m(P, Q)$, is the subcomplex of $\mathrm{sd}^{\,m} Q$ formed by the faces that intersect $\mathrm{sd}^{\,m} P$. If m is a positive integer, $N^m(P, Q)$ is a simplicial complex, even if P and Q are not simplicial.

Let C be a polytopal complex. The *deletion* $C - D$ of a subcomplex D from C is the subcomplex of C of faces not containing D. If some face τ of C is contained in precisely one other face σ of C, we say that τ is a *free* face of σ, and we call the pair (σ, Σ) a *collapse pair* of C. Not all complexes have free faces. An *elementary collapse* is the deletion from C of a free face σ. We denote the elementary collapse by $C \searrow_e C - (\sigma, \Sigma)$, to stress that the face Σ containing σ has also been removed. Given two simplicial complexes C, D, we say that C *collapses onto* D, and write $C \searrow D$, if C can be transformed into D via a sequence of elementary collapses. A complex C is *collapsible* if it collapses onto a single vertex.

[1] The reason why we introduce also polytopal complexes in S^d, and not just in $\mathbb{R}^d$, is that we will need them for the definition of vertex links; see below.

Let P be a pure polytopal complex with N facets. If $\dim P = 0$, a *shelling* of P is any ordering of its vertices. If $\dim P = d > 0$, a *shelling* for P is an ordering $F_1, \ldots, F_N$ of its facets such that for each $i \in \{2, \ldots, N\}$ the complex $F_i \cap \bigcup_{j=1}^{i-1} F_j$ is pure $(d-1)$-dimensional, and is a beginning segment for a shelling of the boundary ∂F_i of F_i. Shellings do not always exist; if P has a shelling, P is called *shellable*. For simplicial complexes, the definition above can be shortened: Since F_i is a d-simplex, any pure $(d-1)$-subcomplex of ∂F_i is automatically the beginning segment for a shelling of ∂F_i.

Let P be a polytopal complex in $\mathbb{R}^k$ (resp. S^k), and let σ be face of P. The *star* of σ in P, denoted by $\mathrm{Star}\,(\sigma, P)$, is the subcomplex of P formed by the faces containing σ. We define the *link* of σ as follows (cf. [9]): Let x be a point in the relative interior of σ. Let N_x denote the subspace of the tangent space at x in $\mathbb{R}^k$ (resp. S^k) spanned by rays orthogonal to σ. The subspace of unit speed geodesics in N_x is isometric to the unit sphere of dimension $k-s-1$, where $s := \dim \sigma$. The intersection of P with the rays in S^{k-s-1} forms a polytopal complex in S^{k-s-1}, called $\mathrm{Link}\,(\sigma, P)$. Combinatorially, the face poset of $\mathrm{Link}\,(\sigma, P)$ is the upper ideal of the element σ in the face poset of P. In case P is a simplicial complex, and σ is a vertex of P, then $\mathrm{Link}\,(\sigma, P)$ is combinatorially equivalent to $\mathrm{Star}\,(\sigma, P) - \sigma$.

A *PL d-ball* is any simplicial complex that has a subdivision combinatorially equivalent to some subdivision of the d-simplex. Similarly, a *PL d-sphere* is a simplicial complex that shares a subdivision with the boundary of the $(d+1)$-simplex. Every d-sphere is PL if and only if the removal of a facet makes it a PL d-ball. Now, let P be a polytopal complex such that the underlying space of P is homeomorphic to a manifold, different than a sphere or a ball. P is called a *PL manifold* if for each vertex v of P the complex $\mathrm{Link}\,(v, P)$ is either a PL sphere or a PL ball. Equivalently, P is PL if for every face σ the complex $\mathrm{Star}\,(\sigma, P)$ is a PL ball.

3 Barycentric subdivisions and the Zeeman conjecture

In this section we show that whatever becomes collapsible after some subdivision, becomes collapsible also after a finite number of *barycentric* subdivisions (Theorem 3.4). Later, we use this to revisit the Zeeman conjecture (Proposition 3.7) and a result by Dierker and by Lickorish (Theorem 3.21).

Let us start by recalling a basic fact in PL topology:

Lemma 3.1 (Whitehead, cf. Zeeman [30, Lemma 13]). *Let B be a PL $(d-1)$-ball in the boundary of a PL d-ball A. For some integer m, some subdivision A' of A collapses onto $\mathrm{sd}^m B$.*

In the particular case where B is a single $(d-1)$-simplex in ∂A, the complex $\mathrm{sd}^m B$ is collapsible. This way one obtains as an immediate corollary of Lemma 3.1 that every PL ball A has a subdivision A' that is collapsible (because it collapses to a collapsible subcomplex).

As Zeeman pointed out, the subdivision A' constructed in [30] need not be barycentric; and in fact, it might not even be stellar. However, with some more technical effort, Glaser showed that one could find another one that is stellar.

Lemma 3.2 (Glaser [16, Theorem III.6]). *Let B be a PL $(d-1)$-ball in the boundary of a PL d-ball A. For some integer m, some stellar subdivision A' of A collapses onto the induced stellar subdivision of B.*

If B is a $(d-1)$-simplex in ∂A, any stellar subdivision of B is collapsible, so Lemma 3.2 immediately implies that every PL ball has a collapsible *stellar* subdivision. We wish to improve this further, by showing that every PL ball has a collapsible *iterated barycentric* subdivision. For this, we need a recent result from the paper [1]:

Theorem 3.3 ([1, Theorem 3.43]). *Let C be a d-complex that is convex in some $\mathbb{R}^k$. Then $\mathrm{sd}^{\,d-2}C$ is endocollapsible.*

Theorem 3.4. *Let C, D be polytopal complexes such that $D \subset C$. Suppose that some subdivision of C collapses simplicially onto some subdivision of D. For m large enough, $\mathrm{sd}^{\,m}\, C \searrow \mathrm{sd}^{\,m}\, D$.*

Proof. Let C', D' be subdivisions of C and D, respectively, such that C' collapses to D'. Let us choose k large enough, so that $\mathrm{sd}^{\,k}C$ is a refinement of C'. For brevity, set $\mathcal{X} := \mathrm{sd}^{\,d-2}(\mathrm{sd}^{\,k}C)$.

Let (σ', Σ') be any collapse pair of C'. Let (δ, Δ) be a pair of faces of $\mathcal{X} = \mathrm{sd}^{\,k+d-2}C$, such that δ is a facet of $\mathcal{R}(\mathcal{X}, |\sigma'|)$, Δ is a facet of $\mathcal{R}(\mathcal{X}, |\Sigma'|)$, and δ is a codimension-one face of Δ. It is easy to see that (δ, Δ) is a collapse pair of $\mathcal{X}$. By Theorem 3.3, $\mathcal{R}(\mathcal{X}, |\Sigma'|) - \Delta$ collapses onto $\mathcal{R}(\mathcal{X}, |\partial\Sigma'|)$). Thus, $\mathcal{X} - \Delta$ collapses onto $\mathcal{R}(\mathcal{X}, |\mathcal{X}| - \mathrm{relint}\,|\Sigma'|)$, or equivalently, $\mathcal{X} - (\delta, \Delta) \searrow \mathcal{R}(\mathcal{X}, |\mathcal{X}| - \mathrm{relint}\,|\Sigma'| - \mathrm{relint}\,|\delta|)$. Now, $\mathcal{R}(\mathcal{X}, |\sigma'|) - \delta$ collapses onto $\mathcal{R}(\mathcal{X}, |\partial\sigma'|)$ by Theorem 3.3, and thus $\mathcal{X} - (\delta, \Delta) \searrow \mathcal{R}(\mathcal{X}, |\mathcal{X}| - \mathrm{relint}\,|\Sigma'| - \mathrm{relint}\,|\sigma'|)$. Thus, if C' can be collapsed onto $C' - (\sigma', \Sigma')$, then $\mathcal{X} \searrow_e \mathcal{X} - (\delta, \Delta)$ and

$$\mathcal{X} - (\delta, \Delta) \;\searrow\; \mathcal{R}(\mathcal{X}, |\mathcal{X}| - \mathrm{relint}\,|\Sigma'| - \mathrm{relint}\,|\sigma'|) \;=\; \mathcal{R}(\mathcal{X}, |C' - (\sigma', \Sigma')|).$$

Repeating this for all collapse pairs involved in the collapse of C' onto D' (in their order), we see that $\mathcal{X}$ collapses to the induced barycentric subdivision of D, as desired. ☐

Corollary 3.5. *Let B be a PL $(d-1)$-ball in the boundary of some PL d-ball A. For m large, $\mathrm{sd}^{\,m}\, A$ collapses onto $\mathrm{sd}^{\,m}\, B$. In particular, every PL ball becomes collapsible after suitably many barycentric subdivisions.*

Proof. Since A and B are PL, some subdivision of A collapses onto some subdivision of B [30, Lemma 13]. The first claim follows then by Theorem 3.4. The second claim follows from the first one, in the particular case where B is a single $(d-1)$-simplex in ∂A. ☐

Remark 3.6. The converse of Corollary 3.5 does not hold. The cone $v * S$ over a non-PL 5-sphere S is a collapsible ball. In particular, after a few barycentric subdivisions, it will collapse onto some boundary facet. However, $v * S$ is not PL.

3.1 A simplicial approach to the Zeeman conjecture

The *Zeeman conjecture* claims that the product of any contractible 2-dimensional complex C with the interval $I = [0, 1]$ is "collapsible in the PL category". This expression (not very common nowadays) should not be confused with "collapsible as polytopal complex", nor with "simplicially collapsible after a barycentric subdivision". What Zeeman claimed is something weaker, namely, that *some* subdivision of $C \times I$ should be simplicially collapsible. In view of Main Theorem 2, this open conjecture can now be rephrased in terms of barycentric subdivisions.

Proposition 3.7. *The Zeeman conjecture is equivalent to the following conjecture: For any contractible 2-complex C, there is an m such that $\mathrm{sd}^{\,m}\,(C \times I)$ is collapsible. One can prove the following weaker statement: For any contractible 2-complex C, there is an m such that $\mathrm{sd}^{\,m}\,(C \times I^5)$ is collapsible.*

Proof. The first claim follows from Theorem 3.4. As for the second claim, Cohen showed in 1975 that if the 4-dimensional Poincaré conjecture is true, then, for any contractible 2-complex C, the product $C \times I^5$ is collapsible "in the PL category", that is, it admits a collapsible PL simplicial subdivision [11, Corollary 3]. The conclusion follows then from Theorem 3.4 and Freedman's 1982 proof of the 4-dimensional Poincaré conjecture [14]. ☐

Our next goal is to show that the number of subdivisions needed, in order to achieve collapsibility, can be arbitrarily high.

Lemma 3.8. *Given any two shellable complexes A and B, the join $A * B$ is shellable.*

Proof. If (A_i) is a shelling sequence (cf. Definition 4.1) for A, and (B_i) is a shelling sequence for B, then the $(A_i * B_j)$'s, with the lexicographic order on the pairs (i, j), form a shelling sequence for $A * B$. □

Lemma 3.9. *Let B be any 3-ball. Let m, n be two non-negative integers. The link of every proper face in $\mathrm{sd}^m(B \times I^n)$ is shellable.*

Proof. Since B is three-dimensional, each link in B is a sphere or a ball of dimension ≤ 2, and therefore shellable. For brevity, set $A := B \times I^n$ and $A' := \mathrm{sd}^m A$. For any face τ of A, there is a face σ in B such then $\mathrm{Link}(\tau, A)$ is obtained from $\mathrm{Link}(\sigma, B)$ by coning. Since coning preserves shellability, we obtain that $\mathrm{Link}(\tau, A)$ is shellable. So also every link in A is shellable. Now, for any face τ' of A', there is a face τ in A such that $\mathrm{Link}(\tau', A')$ can be obtained from $\mathrm{Link}(\tau, A)$ via iterated barycentric subdivisions and joins with polytope boundaries, where the polytopes are faces of A. Polytope boundaries are shellable. Joins and barycentric subdivisions preserve shellability, cf. Lemma 3.8. Hence every link in A' is shellable. □

Definition 3.10 (cf. [5])**.** A triangulation C of a d-manifold with non-empty (resp. empty) boundary is called *endo-collapsible* if C minus a d-face collapses onto ∂C (resp. onto a vertex).

Lemma 3.11 ([5, Corollary 3.21])**.** *Let B be a collapsible PL d-ball. If $\mathrm{sd}\,\mathrm{Link}\,\sigma$ is endocollapsible for every proper face σ, then $\mathrm{sd}\,B$ is also endocollapsible.*

Lemma 3.12 ([5, Theorem 3.1])**.** *All shellable (pseudo)manifolds are endocollapsible.*

Lemma 3.13 ([5, Theorem 3.15])**.** *Let M be an endocollapsible PL d-manifold. Let L be a subcomplex of M, with $\dim L = l \leq d - 2$, such that all facets of L lie in the interior of M. Then the homotopy group $\pi_{d-l-1}(|M| - |L|)$ has a presentation with (at most) $f_l(L)$ generators.*

Theorem 3.14. *For any $m, n \in \mathbb{N}$, there is a 3-ball B such that $\mathrm{sd}^m(B \times I^n)$ is not collapsible.*

Proof. We prove the claim in two parts:
(1) we prove that if $\mathrm{sd}^m(B \times I^n)$ is collapsible, then $\mathrm{sd}^{m+1}(B \times I^n)$ is endocollapsible;
(2) we construct a 3-ball B such that $\mathrm{sd}^{m+1}(B \times I^n)$ is not endocollapsible.
The conclusion follows immediately by combining (1) and (2). So, let us prove these claims.

(1) Via Lemma 3.11, it suffices to show that the link of every proper face in $\mathrm{sd}^{m+1}(B \times I^n)$ is endo-collapsible. In Lemma 3.9 we showed that any such link is shellable; by Lemma 3.12, shellability implies endo-collapsibility.

(2) Let $N = N(m, n)$ be the number of facets of the complex $\mathrm{sd}^{m+1} I^{n+1}$. Let B be a simplicial 3-ball with a 3-edge subcomplex K, isotopic to the connected sum of $3N$ trefoil knots. (For how to construct it, see [6].) Then any presentation of the group $\pi_1(|B| - |K|)$ must have at least $3N + 1$ generators, cf. Goodrick [17]. Now, set $M := \mathrm{sd}^{m+1}(B \times I^n)$ and $L := \mathrm{sd}^{m+1}(K \times I^n)$. This L is $(n + 1)$-dimensional, and has exactly $3N$ facets, while M is $(n + 3)$-dimensional. Clearly $|B| - |K|$ is a deformation retract of $|M| - |L|$, so the homotopy groups of these two spaces are the same. In particular, any presentation of $\pi_1(|M| - |L|)$ must have at least $3N + 1$ generators. Were M endo-collapsible, by Lemma 3.13 we would obtain a presentation of $\pi_1(|M| - |L|)$ with (at most) $3N$ generators, a contradiction. □

Corollary 3.15. *For every positive integers m, n, there is a contractible 2-dimensional simplicial complex C such that $\mathrm{sd}^m(C \times I^n)$ is not collapsible.*

Proof. Let B be any 3-ball for which $\mathrm{sd}^m(B \times I^n)$ is not collapsible; for example, the one constructed in Theorem 3.14. Choose a spanning tree T of the dual path of B, and collapse away all tetrahedra of B "along T", as explained in [6, pp. 214–215]. The resulting 2-complex $C = C(B,T)$ is a deformation retract of B and hence contractible. Moreover, if B collapses onto C, $B \times I^n$ collapses polytopally onto $C \times I^n$ [11, p. 254] and $\mathrm{sd}^m(B \times I^n)$ collapses simplicially onto $\mathrm{sd}^m(C \times I^n)$. Were the latter complex collapsible, $\mathrm{sd}^m(B \times I^n)$ would be collapsible as well, a contradiction. □

Remark 3.16. One could give an alternative, non-constructive proof of Corollary 3.15, using the algorithmic undecidability of the word problem. Here is the idea: Were there universal constants m, n such that for every contractible 2-complexes C the complex $\mathrm{sd}^m(C \times I^n)$ is collapsible, then by checking collapsibility of $\mathrm{sd}^m(C \times I^n)$ we would have an algorithm to decide the contractibility of 2-complexes. Since one can prove that no such algorithm can exist, it follows that no such universal constants exist. We preferred to present the knot-theoretic proof of Corollary 3.15 because it is more elementary, and because it shows how to construct the $C_{m,n}$'s explicitly, for any given pair (m, n).

3.2 Products with high-dimensional cubes

In this section, we show that any contractible complex can be made collapsible by taking products with the n-dimensional cube, for n suitably large. Since the proof resembles the proof of Theorem 3.4, we will only sketch the details.

In order to study products with cubes, we start by introducing a special subdivision, which is in some sense a cubical analog of the classical operation of "starring a face" in a simplicial complex.

Definition 3.17. Let C be a polytopal complex in $\mathbb{R}^d$, and let τ be any face of C. Let C' denote the polytopal complex that is obtained from C by
(1) embedding C into $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$,
(2) deleting τ from C,
(3) attaching the complex $\partial(\mathrm{Star}\,(\tau, C)) \times [0,1]$ to C, and finally
(4) attaching $\mathrm{Star}\,(\tau, C) \times \{1\}$.
This C' has a combinatorially equivalent realization as a subdivision $\mathrm{t}\,(\tau, C)$ of C. We say that $\mathrm{t}\,(\tau, C)$ is obtained from C by *tubing the face* τ.

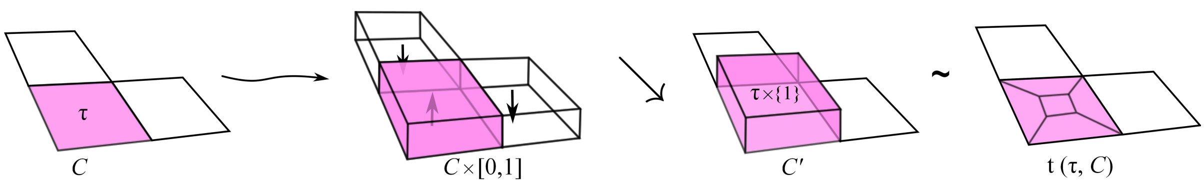

Figure 1: $C \times I$ collapses vertically to C', which is combinatorially equivalent to a subdivision $\mathrm{t}\,(\tau, C)$ of C. We say that this subdivision is obtained by "tubing τ".

The reason why this is the 'right' subdivision to look at, is the following:

Lemma 3.18. *Let C be an arbitrary complex. If $\mathrm{t}\,(\tau, C)$ is obtained from C by tubing some face τ, then the product $C \times [0,1]$ collapses to (a complex combinatorially equivalent to) $\mathrm{t}\,(\tau, C)$.*

Proof. Let σ be a face of C. We perform collapses on $C \times I$ as follows:
– If σ contains τ, we match $\sigma \times \{1\}$ with $\sigma \times [0,1]$. (That is, we match $\sigma \times I$ *downwards*.)

– If σ is not in Star (τ, C), we match $\sigma \times \{1\}$ with $\sigma \times [0,1]$. (That is, we match $\sigma \times I$ *upwards.*)
– If σ is in Star (τ, C), but does not contain τ, we leave it unmatched.
The result is combinatorially equivalent to $\mathrm{t}\,(\tau, C)$: See Figure 1. □

Definition 3.19 (Tubular, tubecentric subdivision)**.** A *tubular subdivision* of a complex C is any subdivision E of C for which there exists a sequence of complexes $C = C_0, C_1, \ldots, C_m = E$, so that each C_{i+1} is obtained from C_i by tubing some face τ.
A *tubecentric subdivision* is a subdivision of a d-complex C obtained by tubing first all d-faces of C, then all $(d-1)$-faces of C, then all $(d-2)$-faces, and so on.

Tubular and tubecentric subdivisions mimic the classical notions of stellar and derived subdivisions, respectively. (Recall that the barycentric subdivision of a d-complex can be obtained by starring first all d-faces of C, then all $(d-1)$-faces of C, and so on.) In fact, tubecentric subdivisions enjoy several analogous properties:

Lemma 3.20. *Let C be a polytopal complex.*
(1) *For any subdivision E of C, some tubecentric subdivision of C is a subdivision of E as well.*
(2) *If C is convex, C becomes endocollapsible after finitely many tubecentric subdivisions.*

Sketch of proof. Item (1) can be seen by adapting the proof of [30, Chapter 1, Lemma 4] to tubular subdivisions. The proof of item (2) is analogous to the proof of [1, Theorem 3.43]. □

We are now ready to show that contractible complexes can be made collapsible by taking products with high-dimensional cubes. Recall that two complexes are called *simple homotopy equivalent* if there exists a sequence of PL expansions and PL collapses that transforms one complex into the other, cf. Zeeman [30].

Theorem 3.21. *If two polytopal complexes C, D are simple homotopy equivalent, there exists an integer n such that $C \times I^n$ collapses to C and to a subdivision of D.*

Proof. Let C and D be two complexes that are simple homotopy equivalent. By a result of Cohen [11, Corollary 5], the complex $E := C \times I^{\max(2\dim C+2,\,7)}$ has a subdivision E' that collapses to a complex PL-homeomorphic to D. By Lemma 3.20, item (1), after performing a few consecutive tubular subdivisions on E, one obtains a polytopal complex E'' that is a subdivision of E' as well. Say that E'' is obtained from E via l tubing steps; by Lemma 3.18, this implies that $E \times I^l \searrow E''$. Now, using Lemma 3.20, item (2), and the same proof of Theorem 3.4, one can show that some iterated tubecentric subdivision F of E'' collapses to a complex PL-homeomorphic to D. If F is obtained via m tubing steps from E'', by Lemma 3.18 we have $E'' \times I^m \searrow F$. Also, from $E \times I^l \searrow E''$ it follows that $E \times I^{m+l} \searrow E'' \times I^m$. In conclusion, we see that $C \times I^{\max(2d+2,7)+m+l} = E \times I^{m+l} \searrow E'' \times I^m \searrow F$, which collapses to (a complex combinatorially equivalent to) a subdivision of D. Clearly, $C \times I^n$ collapses to C for all n. □

Corollary 3.22. *For any contractible complex C, there is an integer n for which $C \times I^n$ is collapsible.*

Proof. Every contractible complex is simple homotopy equivalent to a point [28]. So, choose a vertex v of C, and apply Theorem 3.21 to $D := \{v\}$. □

A stronger notion than collapsibility, called non-evasiveness, was introduced for *simplicial* complexes in [19] in connection with the Aanderaa–Karp–Rosenberg conjecture. One can easily extend the notion to *polytopal* complexes, as follows: A polytopal d-complex C is called *non-evasive* if either $d = 0$ and C is a single point, or $d > 0$ and there is a vertex v of C so that both Link (v, C) and $C - v$ are non-evasive. This property is maintained under several combinatorial constructions:

Proposition 3.23 (Welker [27]). (i) *All non-evasive complexes are collapsible.*
(ii) *The barycentric subdivision of every collapsible complex is non-evasive.*
(iii) *The product of any two non-evasive complexes is non-evasive.*

Welker [27, Open Problem 2] asked whether the converse of item (iii) holds; more precisely, he asked whether the non-evasiveness of the product $B \times C$ of two order complexes B, C implies that both B and C are non-evasive. Using products with intervals, we can provide a negative answer to Welker's question.

Proposition 3.24. *There exists an evasive (order) complex C such that $C \times I$ is non-evasive.*

Proof. Consider the simplicial complex C of Figure 2, below.

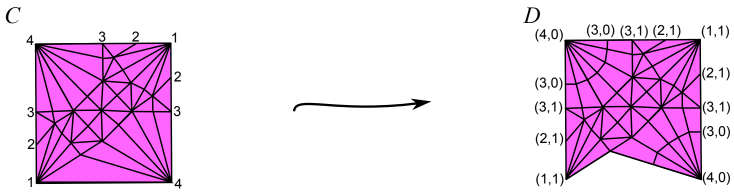

Figure 2: The complex C on the left (marked vertices are identified) is collapsible and evasive. However, $C \times I$ is non-evasive: In fact, by deleting a few vertices with non-evasive links, $C \times I$ can be reduced to the complex D on the right, which is non-evasive.

This C is collapsible (starting from the bottom edge), but it cannot be non-evasive, because all of its vertex links retract to 1-spheres, hence they cannot be collapsible; compare [3, Example 5.4]. We leave it to the reader to check that C is an order complex. Consider now the complex $C \times I$, in which every vertex v of C corresponds to two copies $(v, 0)$ and $(v, 1)$. Let us start deleting from $C \times I$ all the vertices of the type $(w, 0)$, such that w is a vertex not adjacent to 4 in C. Then, let us delete $(4, 1)$ and $(1, 0)$. The remaining complex is combinatorially isomorphic to the complex D of Figure 2. To reduce D further, we delete the vertices of D that are not $(1, 1)$, $(2, 1)$, $(3, 1)$ or $(3, 0)$ or $(4, 0)$ first. The remainder is then a tree, and all trees are non-evasive. To conclude, one should verify that every time we deleted a vertex, its link was non-evasive; this verification is straightforward and left to the reader. □

Inspired by Proposition 3.24 and Corollary 3.22, we end this section with a question:

Problem 1. *Given any contractible complex C, is there an integer n so that $C \times I^n$ is non-evasive?*

An affirmative answer to Problem 1 would imply the existence of vertex-homogeneous non-evasive polytopal complexes different than balls.

4 Shellability of convex complexes

In the previous section, we saw how barycentric subdivisions suffice to make any PL ball collapsible. That said, collapsibility is a too weak notion to characterize the PL property, basically because collapsible balls need not be PL (Remark 3.6). In fact, collapsibility is even too weak to characterize balls: Collapsible manifolds might be of different homeomorphism types [1]. For these reasons, we shift our focus to the stronger notion of shellability. All shellable contractible complexes are collapsible; moreover, all shellable contractible manifolds are PL balls. It remains to be seen whether Corollary 3.5 can be adapted, namely, whether barycentric subdivisions eventually suffice to make any PL ball shellable. This is the goal of the present section.

Recall that convex 3-complexes are always collapsible, but they are not always shellable, as highlighted by Rudin's example [26]. Indeed, we will show that the $(d-2)$-nd barycentric subdivision of any convex d-complex is shellable (Theorem 4.17). In particular, we will obtain that Rudin's ball becomes shellable after a single barycentric subdivision.

Since the proof is quite technical, we need to introduce a few definitions first.

Definition 4.1 (Shelling sequences)**.** Let C be a simplicial d-complex with N facets. Assume that $d(N-1) > 0$. A *shelling step* is the removal from a pure simplicial d-complex C of a single d-face whose intersection with the remaining complex is pure $(d-1)$-dimensional and shellable. Given two pure simplicial d-complexes C and D, we say that "*C shells to D*", and write $C \searrow_{\mathrm{S}} D$, if there is a sequence of shelling steps which leads from C to D. The intermediate complexes between C and D form a sequence of subcomplexes (C_i), which we call *shelling sequence.* By definition, C_{i+1} is a subcomplex of C_i obtainable from C_i via a single shelling step. Clearly, a complex C is shellable if and only if C shells to some facet Δ.

Definition 4.2 (D-conforming)**.** Let C be a simplicial d-complex, and let D be a subcomplex. Any shelling sequence (C_i) for C induces a sequence of subcomplexes of D, defined by $D_i := \mathcal{R}(C_i, |D|)$. Clearly $D_{i+1} \subset D_i$. If all the D_i are pure, then the shelling (C_i) of C is called *D-conforming.* If C is a simplicial complex in $\mathbb{R}^k$ (resp. S^k), and A is a subset of $\mathbb{R}^k$ (resp. S^k), then a shelling of C is called *A-conforming* if it is $\mathcal{R}(C, A)$ conforming.

Example 4.3. Let B and C be two simplicial PL d-balls that intersect at a $(d-1)$-ball D, in the boundary of both. Assume that C is shellable. If the shelling sequence for C is D-conforming, then $B \cup C \searrow_S D$. This is not always the case: For example, Rudin's ball is the union of two shellable 3-balls B and C, glued together at a 2-ball D in their boundary. However, any shelling sequence of B (or of C) is not D-conforming: In fact, Rudin's ball is not shellable.

Lemma 4.4. *If $B \searrow_{\mathrm{S}} C$, then $\mathrm{sd}^{\,m} B \searrow_{\mathrm{S}} \mathrm{sd}^{\,m} C$ for all m.*

Definition 4.5 (Stacked)**.** Let Δ be a $(d-1)$-simplex. Let $P = \Delta \times I$. A *stacked* subdivision of P is any subdivision that has no $(d-2)$-faces in the interior of P, and that does not subdivide the two bases $\Delta \times \{0\}$ and $\Delta \times \{1\}$ of the prism. Such a subdivision is combinatorially a path of d-simplices, so it shells to any of its d-faces (exactly like its dual graph, which is a path, can be shelled to any of its vertices, by recursively deleting leaves).

Lemma 4.6. *Let C and D be two simplicial d-complexes with $C \searrow_{\mathrm{S}} D$. Let C' be some subdivision of $C \times I$ that is stacked when restricted to any facet of $C \times I$. Then C' shells to $\mathcal{R}(C', D \times I)$. In particular, if D is a facet of C, then C' is shellable.*

Proof. It is enough to prove the case where $C \searrow_{\mathrm{S}} D$ is a single shelling step. Let Δ denote the facet removed in the step. By assumption, $\mathcal{R}(C', \Delta \times I)$ is a stacked subdivision: In fact, it is a path of $(d+1)$-simplices, so in particular it shells to each of its facets. Now, choose a facet F of $\mathcal{R}(C', \Delta \times I)$ that intersects $D \times I$ in a d-ball, and choose a shelling sequence (C_i) of $\mathcal{R}(C', \Delta \times I)$ to F. Then, using Example 4.3 we can use this shelling to shell C' to $\mathcal{R}(C', D \cup F)$, because the intersection of the (C_i) with $D \times I$ is always pure. Finally, we remove F. $\square$

Lemma 4.7. *Let C be a simplicial complex. Let v be a new vertex. Let m be a positive integer. If $\mathrm{sd}^{\,m} C$ is shellable, then $\mathrm{sd}^{\,m}(C * v)$ is shellable, and the shelling can be chosen to be $(\mathrm{sd}^{\,m} C)$-conforming.*

Proof. The complex $\mathrm{sd}^{\,m}(C * v) - v$ has a combinatorially equivalent realization as a subdivision of $\mathrm{sd}^{\,m}(C) \times I$ that is stacked on every facet of $\mathrm{sd}^{\,m}(C) \times I$. The conclusion follows by applying Lemma 4.6 to $\mathrm{sd}^{\,m}(C * v) - v$. $\square$

Now, let C be a simplicial complex in the sphere S^d. Let U be a hemisphere of S^d whose boundary $E = \partial U$ does not contain any vertex of C. Assume all facets of C intersect E in their relative interior. Let U_C be the subcomplex of faces of C in U; let L_C denote the subcomplex of faces of C in the complementary hemisphere. Let E_C be the polytopal complex that arises as the intersection of E with the simplices of C. Let F be a subcomplex of C every single of facets intersect E, and let E_F be the complex induced by the intersection of F with E.

Lemma 4.8. *With the previous notation, if* $\mathrm{sd}^{\,m} E_C \searrow_S \mathrm{sd}^{\,m-1} N^1(E_F, E_C)$ *for some* $m \geq 2$, *then*

$$\mathrm{sd}^{\,m} C \searrow_{\mathrm{S}} \mathrm{sd}^{\,m-1} N^1(F, C) \cup \mathrm{sd}^{\,m-2} N^2(U_C, C) \cup \mathrm{sd}^{\,m-1} N^1(L_C, C).$$

Moreover, if the former shelling is M-conforming, for some subcomplex M of C all whose facets intersect E, the latter shelling can be chosen to be M-conforming.

Proof. The complex

$$\mathrm{sd}^{\,m} C \;-\; \mathrm{sd}^{\,m-2} N^2(U_C, C) \cup \mathrm{sd}^{\,m-1} N^1(L_C, C)$$

is combinatorially equivalent to

$$\mathrm{sd}^{\,m}(E_C \times I) \;-\; \mathrm{sd}^{\,m-2} N^2(E_C \times \{1\}, E_C \times I) \;-\; \mathrm{sd}^{\,m-1} N^1(E_C \times \{0\}, E_C \times I).$$

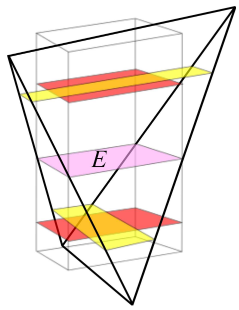

Figure 3: After removing the subcomplexes $\mathrm{sd}^{\,m-2} N^2(U_C, C) \cup \mathrm{sd}^{\,m-1} N^1(L_C, C)$ from $\mathrm{sd}^{\,m} C$ (indicated in yellow) and $\mathrm{sd}^{\,m-2} N^2(E_C \times \{1\}, E_C \times I) \cup \mathrm{sd}^{\,m-1} N^1(E_C \times \{0\}, E_C \times I)$ from $\mathrm{sd}^{\,m}(E_C \times I)$ (indicated in red), the remaining complexes are combinatorially isomorphic.

$\mathrm{sd}^{\,m}(E_C \times I)$ is combinatorially equivalent to a subdivision of $\mathrm{sd}^{\,m} E_C \times I$ that is stacked on every prism. Since $\mathrm{sd}^{\,m} E_C \times \{0\} \searrow_{\mathrm{S}} \mathrm{sd}^{\,m-1} N^1(E_F, E_C) \times \{0\}$, Lemma 4.6 applies. However, instead of shelling the prisms away completely, we leave the desired facets in the neighborhood of $E_C \times \{1\}$ and $E_C \times \{0\}$, that is, we do not remove the facets in $\mathrm{sd}^{\,m-2} N^2(E_C \times \{1\}, E_C \times I)$ and $\mathrm{sd}^{\,m-1} N^1(E_C \times \{0\}, E_C \times I)$. This shells $\mathrm{sd}^{\,m}(E_C \times I)$ to

$$\mathrm{sd}^{\,m-2} N^2(E_C \times \{1\}, E_C \times I) \;\cup\; \mathrm{sd}^{\,m-1} N^1(E_C \times \{0\}, E_C \times I) \;\cup\; \mathrm{sd}^{\,m-1} N^1(E_F \times I, E_C \times I).$$

If, for every of the prisms, we choose the starting facet to intersect $(E \cap M) \times I$ in a pure $(d-1)$-ball, and then shell towards the bases of the prism, the resulting shelling is $(E \cap M) \times I$-conforming. Finally, using the combinatorial isomorphism between

$$\mathrm{sd}^{\,m}(E_C \times I) \;-\; \mathrm{sd}^{\,m-2} N^2(E_C \times \{1\}, E_C \times I) \;-\; \mathrm{sd}^{\,m-1} N^1(E_C \times \{0\}, E_C \times I)$$

and

$$\mathrm{sd}^{\,m} C \;-\; \mathrm{sd}^{\,m-2} N^2(U_C, C) \cup \mathrm{sd}^{\,m-1} N^1(L_C, C),$$

the shelling of the former induces the desired shelling on the latter. $\square$

Definition 4.9 (Pseudosimplex)**.** A (d-dimensional) *pseudosimplex* Δ in S^d is the intersection of $d+1$ or less hemispheres whose centers lie in convex position. The boundary $\partial\Delta$ has a natural decompositon into pseudosimplices, which we will call *faces*. In the special case when Δ is defined by one hemisphere, we call $\partial\Delta$ an *equator*. If p is a point, we denote by L_p the hemisphere with center p, and by E_p the equator ∂L_p.

Definition 4.10 (p-spanning boundary complexes)**.** Let P be a pseudosimplex in S^d (resp. a simplex in $\mathbb{R}^d$). Let M be a pure subcomplex of ∂P, either empty or $(d-1)$-dimensional. Let p be a point in S^d (resp. a vector in $\mathbb{R}^d$). We say that M is *p-spanning* if the hemisphere centered at p contains the intersection of all facets of M or does not intersect the hemisphere at all (resp. if the unique local minimum of the scalar product $\langle p, -\rangle$ on M is contained in the intersection of all facets of M).

Every pure $(d-1)$-complex $M \subset \partial\Delta$ is p-spanning for some p. This can easily be seen by choosing a p for which $\langle p, -\rangle$ attains its minimum on Δ in τ. (Recall that every face of a simplex maximizes some linear functional, cf. [32].) More generally, we can state the following. Let P be a pseudosimplex in S^d, or a simplex in $\mathbb{R}^d$. If x is a point in ∂P, and M is a p-spanning subset of P, we denote by T_x the space of unit speed geodesics emanating from x. By T_xP resp. T_xM we denote the subset of T_x formed by the geodesics that lie entirely in P resp. in M.

Lemma 4.11. *Let P be a pseudosimplex in S^d, or a simplex in $\mathbb{R}^d$.*
(i) *T_xP is a pseudosimplex, and T_xM is a set of facets of T_xP.*
(ii) *If P is a simplex in $\mathbb{R}^d$, and ϱ is the geodesic ray from x in direction p, then T_xM is a ϱ-spanning subset of T_xP.*

Proof. Item (i) follows from the definition of pseudosimplex. In fact, P is defined by at most $d+1$ hemispheres, whose centers lie in convex positions; hence, at most d of these hemispheres contain x in the boundary. Thus, P_x is a subset of S^{d-1}, defined by at most d hemispheres. The general position of their centers is obvious, since they correspond to the geodesic rays from x to the centers of the hemispheres defining P that intersect x in the boundary. Item (ii) is a direct consequence of the definition of p-spanning. $\square$

Definition 4.12 (General position, k-split-shellable)**.** Let C be a simplicial complex in the sphere S^d, that is convex with respect to the standard intrinsic metric on S^d (resp. $\mathbb{R}^d$). (This means that for any two points of C, a shortest arc between them is completely contained in $|C|$.) A hemisphere (resp. a halfspace) H of S^d (resp. $\mathbb{R}^d$) intersecting C is *in general position with respect to C* if no vertex of C lies in ∂H. If k is a positive integer, we say that C is *k-split-shellable* if for every hemisphere (resp. any halfspace) H in general position with respect to C, the shelling of $\mathrm{sd}^k C$ can be chosen so that $\mathrm{sd}^k C$ shells to $\mathrm{sd}^{k-1} N^1(\mathcal{R}(C,H),C)$, which then shells to some facet.

Definition 4.13 (Strongly k-split-shellable)**.** Let C be a refinement of some pseudosimplex Δ in S^d or of a simplex in $\mathbb{R}^d$. We call C *strongly shellable* if, for every $(d-1)$-boundary complex M in $\partial\Delta$ that is p-spanning (with respect to some p), there is a shelling of C that is M-conforming. Let C be a refinement of some pseudosimplex P in S^d (resp. of a simplex in $\mathbb{R}^d$). We call C *strongly k-split-shellable* if C is k-split-shellable, and the shelling of definition 4.15 can be chosen to be $\mathcal{R}(\mathrm{sd}^k C, |D|)$-conforming, for any p-spanning subcomplex D of ∂P, where p is the center of the hemisphere H (resp. where p is the exterior normal to ∂H).

Example 4.14. If $B \subsetneq D$ are two simplicial 2-balls, we can shell D to B by recursively removing a facet of D that intersects ∂D in a 1-face, and is not in B. Such a facet always exists: In fact, if B intersected the whole boundary of D, then B and D would coincide, cf. [2, Lemma 2.3]. Now, consider any subdivision C of S^2. Let x be a point of S^2 in general position. Let L_x and U_x be the hemispheres with center x and $-x$, respectively. After the removal of a facet in the hemisphere U_x from $\operatorname{sd} C$, the remaining complex C' is a 2-ball, containing $N^1(\mathcal{R}(C, L_x))$ as a subset. By the previous observation, we can shell $\operatorname{sd} C$ to $N^1(\mathcal{R}(C, L_x))$. Subsequently, we can shell away the 2-ball $N^1(\mathcal{R}(C, L_x))$, which implies that C is 1-split shellable.

Definition 4.15 (E-splitting subdivisions)**.** Let P be a polytopal complex in S^d. Let E be an equator of S^d. The *E-splitting subdivision of* P is the (realization of a) derived subdivision where vertices are chosen so that:
- they coincide with the barycenters of those faces whose relative interior does not intersect E;
- they lie at the barycenter of the intersection $E \cap \tau$, for any face τ of P whose relative interior intersects the equator E.

Definition 4.16 (Lower link)**.** Let v be a vertex of a simplicial complex T in $\mathbb{R}^d$, and let y be a generic vector in $\mathbb{R}^d$. The *lower link of T at v with respect to y*, denoted by $\mathrm{Lowlink}^y(v, T)$, is the subcomplex of $\mathrm{Link}\,(v, T)$ induced by vertices w of T with $\langle y, w\rangle < \langle y, v\rangle$.

Theorem 4.17. *Let d be a positive integer. Set $d^- = \max(1, d-1)$, $d^= = \max(1, d-2)$.*
(I) *If C is any subdivision of a convex set P in S^d, then C is d^--split-shellable.*
If in addition P is a pseudosimplex, then C is also strongly d^--split-shellable.
(II) *If C is any subdivision of a convex set P in $\mathbb{R}^d$, then C is $d^=$-split shellable.*
If in addition P is a simplex, then C is also strongly $d^=$-split shellable.

Proof. We prove (I) and (II) simultaneously, by induction on d. The proof is in four parts:
(1) for $d = 1$, we prove (I) and (II) directly;
(2) for $d = 2$, we show that (II) implies (I);
(3) for $d \geq 3$, we assume that (I) and (II) hold for all complexes of dimension $< d$, and we prove (I) for complexes of dimension d;
(4) for $d \geq 2$, we assume that (I) and (II) hold for complexes of dimension $< d$, and we prove (II) for complexes of dimension d.

Clearly, this suffices to prove the theorem. So, let us prove the four claims above.

(1) Every subdivision of an interval, or of the 1-sphere, is shellable; moreover, any shelling is boundary-conforming, which proves strong shellability. Split-shellability can be proven by removing the vertices according to their distance from x. (In case two vertices have equal distance from x, we can wiggle the complex a bit.)

(2) For $d = 2$, one has $d^- = d^= = 1$. Let C be any triangulation of a convex set in S^2. If C is a subdivision of the whole S^2, it is 1-split shellable by Example 4.14, and we are done. So, let us assume that C is a 2-ball in S^2 (resp. a subdivision of a 2-dimensional pseudosimplex in S^2). Being convex, C is completely contained in a closed hemisphere of S^2. Let us wiggle the vertices of C a bit, so that they all lie in the interior of the hemisphere. The central projection of such hemisphere to $\mathbb{R}^2$ maps C to a subdivision of a convex set in $\mathbb{R}^2$ (resp. to a simplex in $\mathbb{R}^2$). This shows that every convex ball in S^2 is combinatorially equivalent to some convex ball in $\mathbb{R}^2$. Since hemispheres in S^2 project to halfspaces in $\mathbb{R}^2$, split-shellability in the 2-sphere reduces to split-shellability in the plane.

(3) Let us assume $d \geq 3$, so that $d^- = d - 1$ and $d^= = d - 2$. Let C be a subdivision of an arbitrary convex set P in S^d. The special case where P lies in one open hemisphere of S^d, and thus projects to a combinatorially equivalent complex in $\mathbb{R}^d$, is treated in part (4). (In brackets,

we will consider the case where P is a pseudosimplex in S^d, but not a simplex.) Let x be a point of S^d in general position. Let L_x denote the closed hemisphere with center x. Let U_x denote the complementary closed hemisphere. Finally, let E_x denote the common boundary of L_x and U_x. Let y be a generic point in E_x. (If P is a pseudosimplex, we choose y to lie in $E_x \cap P$ in such a way that $E_x \cap M$ is a y-spanning set of facets in the boundary of the pseudosimplex $E_x \cap P$. This can be done by choosing y to be close to circumcenter of $E_x \cap P$, if the latter is a simplex. If instead $E_x \cap P$ is not a simplex, y can be chosen freely.) Let L_y be the closed hemisphere with center y, and let U_y be the complementary closed hemisphere. Let C' be the E_x-splitting subdivision of C, as defined in 4.15. Let us subdivide A into the two complexes

$$A_1 := \mathrm{sd}^{\,d-3} N^1(\mathcal{R}(C', E_x \cap U_y), \mathcal{R}(C', U_x)) \text{ and } A_3 := \mathrm{sd}^{\,d-3} N^1(\mathcal{R}(C', E_x \cap L_y), \mathcal{R}(C', U_x)).$$

Finally, let us define

$$A_2 := \mathrm{sd}^{\,d-3} N^1(\mathcal{R}(C', \mathrm{int} U_x), C') \text{ and } A_4 := \mathrm{sd}^{\,d-2} \mathcal{R}(C', \mathrm{int} L_x).$$

A_4 is combinatorially equivalent to $\mathrm{sd}^{\,d-2} N^1(\mathcal{R}(C, \mathrm{int} L_x), C)$. Our goal is to show that A_4 is shellable, and that

$$\mathrm{sd}^{\,d-2} C' = A_1 \cup A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_3 \cup A_4 \searrow_{\mathrm{S}} A_4.$$

(In case P is a pseudosimplex, we also have to show that the shelling can be chosen to be M-conforming.)

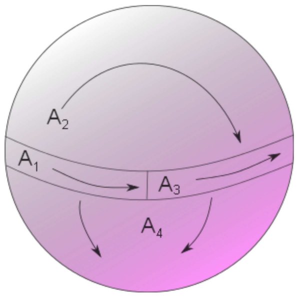

Figure 4: An illustration of the shelling process $\mathrm{sd}^{\,d-2} C' = A_1 \cup A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_3 \cup A_4 \searrow_{\mathrm{S}} A_4$ on the 2-sphere. First, we remove A_1 (using Lemma 4.8), then A_2 (using a linear functional), then A_3 (with the same procedure as the removal of A_1), and finally, we shell away A_4.

Since $\mathcal{R}(C', E_x)$ is a convex complex in S^{d-1}, we can apply the inductive assumption on (II) if $E_x \cap P$ lies in some open hemisphere, and the inductive assumption on (I) otherwise. We have in any case that $\mathrm{sd}^{\,d-2} \mathcal{R}(C', E_x)$ shells to $\mathrm{sd}^{\,d-3} N^1(\mathcal{R}(C', E_x \cap L_y), \mathcal{R}(C', E_x))$. (If P is a d-pseudosimplex, then $\mathcal{R}(C', E_x)$ is a pseudosimplex in S^{d-1}; and the latter shelling can be chosen to be M-conforming, by induction.) By Lemma 4.8 we obtain

$$\mathrm{sd}^{\,d-2} C' = A_1 \cup A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_2 \cup A_3 \cup A_4.$$

Next, let us denote by π_x the central projection of U_x to $\mathbb{R}^{d-1}$, and by $\hat{y}$ a vector in the interior of the ray from $0 = \pi_x x$ to $\pi_x(-y)$. We are going to describe a shelling $A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_3 \cup A_4$ which removes the subcomplexes $\mathrm{sd}^{\,d-3} N^1(v, C')$ of C', in the interior of U_x, according to some special order of the vertices v of C', which we will now explain. Let v' be the vertex of C', in the interior of U_x, that maximizes the interior product $\langle \hat{y}, \pi_x v' \rangle$ over all vertices of C'. Let

D be the complex obtained by removing $\mathrm{sd}^{\,d-3}N^1(v', C')$ from C'. Then the intersection of $\mathrm{sd}^{\,d-3}N^1(v', C')$ with D is combinatorially equivalent to

$$\mathrm{sd}^{\,d-3}N^1(\mathrm{Lowlink}^{\hat{y}}\,(\pi_x v', \pi_x C'), \mathrm{Link}\,(\pi_x v', \pi_x C')).$$

Since the tangential cone at any convex set is convex, $\mathrm{Link}\,(\pi_x v', \pi_x C')$ is a convex subset of S^{d-1}. We can apply the inductive assumption to it, and conclude that $\mathrm{Link}\,(\pi_x v', \pi_x C')$ is $(d-2)$-split-shellable. (If in addition P is a pseudosimplex, so is $\mathrm{Link}\,(\pi_x v', \pi_x C')$, by Lemma 4.11; so by inductive assumption the shelling can be chosen to be M-conforming.) By Lemma 4.7, we can shell $\mathrm{sd}^{\,d-2}\,(\mathrm{Link}\,(\pi_x v', \pi_x C') * \pi_x v')$ to the subcomplex

$$\mathrm{sd}^{\,d-3}\left(N^1(\mathrm{Lowlink}^{\hat{y}}\,(\pi_x v', \pi_x C'), \mathrm{Link}\,(\pi_x v', \pi_x C')) * \pi_x v'\right),$$

which has a shelling that is $N^1(\mathrm{Lowlink}^{\hat{y}}\,(\pi_x v', \pi_x C'), \mathrm{Link}\,(\pi_x v', \pi_x C'))$-conforming. Therefore, as in Example 4.3,

$$\mathrm{sd}^{\,d-3}\,C' - A_1 = A_2 \cup A_3 \cup A_4 \ \searrow_{\mathrm{S}} \ D.$$

Next, let v'' be a vertex of D, in the interior of U_x, that is a vertex of C' as well and that maximizes the interior product $\langle \hat{y}, \pi_x v''\rangle$ over all vertices with such properties. Reasoning as above, we obtain that

$$D \ \searrow_{\mathrm{S}} \ D' := \mathrm{sd}^{\,d-3}\,\mathcal{R}(D, \mathrm{cl}(S^d \setminus \mathrm{sd}^{\,d-3}N^1(v'', C'))).$$

And so on. We can repeat this process until all vertices of C' in U_x are removed. This way, we can shell

$$A_2 \cup A_3 \cup A_4 \searrow_{\mathrm{S}} A_3 \cup A_4.$$

Now, consider $\mathcal{R}(C', E_x \cap L_y)$: This is a convex subset of S^{d-1} (and even a pseudosimplex, if P is.) By the inductive assumption, $\mathrm{sd}^{\,d-2}\mathcal{R}(C', E_x \cap L_y)$ is shellable (with an M-conforming shelling.) Therefore, using Lemma 4.8 as above, we can shell

$$A_3 \cup A_4 \searrow_{\mathrm{S}} A_4$$

(in an M-conforming way).
Finally, let us show that A_4 is shellable. The argument above proves that $\mathrm{sd}^{\,d-2}\mathcal{R}(C', \mathrm{int}U_x)$ is shellable. By replacing x with $-x$, one sees that $A_4 = \mathrm{sd}^{\,d-2}\mathcal{R}(C', \mathrm{int}L_x)$ is shellable.

(4) Let C be any subdivision of a convex set P in $\mathbb{R}^d$. Let H be some halfspace in $\mathbb{R}^d$ with exterior normal x. (If P is a simplex, choose also an x-spanning boundary complex M.) We are going to describe a shelling of $\mathrm{sd}^{\,d^=}C$ that removes the subcomplexes $\mathrm{sd}^{\,d^=-1}N^1(v, C)$ using a special order of the vertices v of C, based on how large the interior product $\langle x, v\rangle$ is.

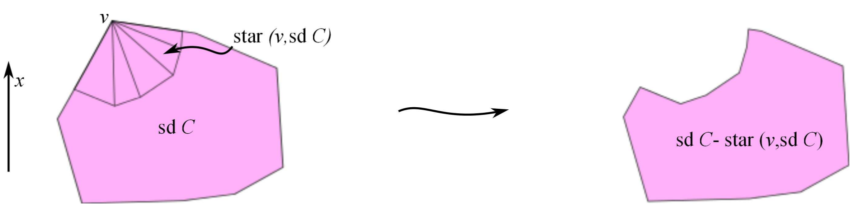

Figure 5: The shelling process for a convex ball in $\mathbb{R}^2$: We delete the vertices v of C from $\mathrm{sd}\,C$, in the order given by how large their interior product with the direction x is.

Let v be the vertex of C maximizing $\langle x, v\rangle$. Let D be the complex obtained by removing $\mathrm{sd}^{\,d^=-1}N^1(v, C)$ from C. The intersection of $\mathrm{sd}^{\,d^=-1}N^1(v, C)$ with D is combinatorially equivalent to $\mathrm{sd}^{\,d^=-1}\,N^1(\mathrm{Lowlink}^x(v, C), \mathrm{Link}\,(v, C))$. Since $\mathrm{Link}\,(v, C)$ is convex (and even a pseudosimplex, if P is a simplex), we can use the inductive assumption to obtain $\mathrm{sd}^{\,d^=}\,\mathrm{Link}\,(v, C) \ \searrow_{\mathrm{S}}$

$\mathrm{sd}^{d^= -1} N^1(\mathrm{Lowlink}^x(v, C), \mathrm{Link}\,(v, C))$. By Lemma 4.7, $\mathrm{sd}^{d^=} (v * \mathrm{Link}\,(v, C)) \searrow_S \mathrm{sd}^{d^= -1} (v * N^1(\mathrm{Lowlink}^x(v, C), \mathrm{Link}\,(v, C)))$ (and this shelling can be chosen to be M-conforming). Since $N^1(\mathrm{Lowlink}^x(v, C), \mathrm{Link}\,(v, C))$ is shellable, using Lemma 4.7 and Example 4.3 as in the shelling of A_2 in part (3), we conclude

$$\mathrm{sd}^{d^=} C \searrow_S D.$$

Next, let v' be a vertex of D that is a vertex of C as well, and that maximizes the interior product with x over all vertices with such properties. Let D' be the complex obtained by removing $\mathrm{sd}^{d^= -1} N^1(v', C)$ from D. Reasoning as above, we obtain that

$$D \searrow_S D'.$$

And so on; eventually, all vertices of C in $(\mathbb{R}^d \setminus H)$ are removed. Let us continue until the complex is reduced to a single d-simplex. This proves the (strong) $d^=$-split-shellability of C. □

Corollary 4.18. *Let C be a subdivision of a convex ball in $\mathbb{R}^d$. Then $\mathrm{sd}^{d-2} C$ is shellable.*

Corollary 4.19. *Let C be a subdivision of a d-simplex Δ. Then for any pure $(d-1)$-subcomplex M of $\partial\Delta$, there is a shelling of $\mathrm{sd}^{d-2} C$ which is $\mathcal{R}(\mathrm{sd}^{d-1} C, M)$-conforming.*

Proof. Remember that every set M of facets of the simplex is p-spanning with respect to some direction. □

Remark 4.20. One might wonder if there is some kind of converse statement to Corollary 4.18: Does every shellable ball become combinatorially equivalent to a convex ball after a limited number of barycentric subdivisions? The answer is negative: For each $d \geq 4$ and for each positive integer m, one can construct a shellable d-ball B_m whose m-th barycentric subdivision has non-shellable boundary [2, Remark 3.11]. In particular, (any complex combinatorially equivalent to) the m-th barycentric subdivision of B_m cannot be convex.

5 A characterization of the PL property via shellability

We have already all the tools to extend Theorem 3.4 to shellability.

Theorem 5.1. *Let C, D be polytopal complexes. Suppose that $C' \searrow_S D'$ for some subdivisions C' and D' of C and D, respectively. Then, for m large enough, $\mathrm{sd}^m C \searrow_S \mathrm{sd}^m D$.*

Proof. As in Theorem 3.4, let us choose m large enough, so that $\mathrm{sd}^m C$ is a subdivision of C'. We set $\mathcal{X} := \mathrm{sd}^{d-2}(\mathrm{sd}^m C)$. Consider a shelling step of C', consisting in the removal of some facet σ from C'. By Corollary 4.19, there is a shelling of $\mathcal{R}(\mathcal{X}, |\sigma|)$ that is $|\sigma| \cap \mathrm{cl}(|C'| \setminus |\sigma|)$-conforming. By Example 4.3,

$$\mathcal{X} \searrow_S \mathcal{R}(\mathcal{X}, \mathrm{cl}(|C'| \setminus |\sigma|)).$$

Repeating this for all shelling steps of C' onto D' (in their order), we see that $\mathcal{X}$ shells to the induced barycentric subdivision of D, as desired. □

In particular, any simplicial complex that is shellable after some subdivision, is also shellable after some iterated barycentric subdivision.

Theorem 5.2. *For a triangulated manifold M, the following are equivalent:*
(1) *M is a PL ball or a PL sphere.*
(2) *For some positive integer m, the m-th barycentric subdivision of M is shellable.*

Proof. The direction (2) $\Rightarrow$ (1) is well known: One can prove by induction that all shellable d-pseudomanifolds, for $d > 0$, are either PL balls or PL spheres. The direction (1) $\Rightarrow$ (2) follows from Theorem 5.1, and the fact that every PL d-ball has some subdivision which is also some iterated barycentric subdivision of the d-simplex (hence shellable). $\square$

The main advantage of iterated barycentric subdivisions over arbitrary stellar subdivisions, is that we know how they look like. Given a complex C, plenty of complexes (in fact, as many as the faces of C) can be obtained from C via a single stellar subdivision. Instead, we can describe $\mathrm{sd}^{\,i} C$ explicitly. In theory, we could try all possible facet orderings of $\mathrm{sd}^{\,i} C$ and check if any of them is a shelling. Therefore, the class S_i of all triangulated d-balls whose i-th barycentric subdivision is shellable, is algorithmically recognizable. In contrast, when $d \geq 5$ the class of PL d-balls is not algorithmically recognizable. However, by Theorem 5.2,

$$S_0 \subset S_1 \subset \ldots \subset S_i \subset \ldots \subsetneq \bigcup_{i=0}^{\infty} S_i = \{\text{PL } d\text{-balls}\}.$$

Another advantage of the characterization given by Theorem 5.2 is that it can be used for enumerating triangulations of spheres. It has been conjectured that, for fixed d, there are only exponentially many combinatorially inequivalent triangulations of S^d with N tetrahedra, cf. [6]. At present, this has been proven only for $d = 2$. Recently the second author and Ziegler have shown that *shellable* triangulations of S^d are exponentially many [6]. Now, the i-th barycentric subdivision of a d-complex with N facets has $C_{i,d} N$ facets, where $C_{i,d}$ depends only on i and d; therefore, what is exponential in $C_{i,d} N$ is also exponential in N. This leads to the following statement:

Proposition 5.3. *For fixed i and d, there are exponentially many triangulations of S^d with shellable i-th barycentric subdivision.*

Thus the family (S_i) described above is a family of subclasses of exponential size that "approximates" the set of all PL d-balls. Combinatorially, the smallest i for which S_i contains B can be interpreted as the distance of B from being shellable.

We conclude by expanding Theorem 5.2 to a characterization of PL manifolds:

Theorem 5.4. *For any triangulated manifold M, the following are equivalent:*
(1) *M is PL.*
(2) *Every vertex link in M becomes shellable after suitably many barycentric subdivisions.*
(3) *For some $m \in \mathbb{N}$, the m-th barycentric subdivision of any vertex link in M is shellable.*
(4) *For some $m \in \mathbb{N}$, all vertex links in $\mathrm{sd}^{\,m} M$ are shellable.*

Proof. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.
(1) $\Leftrightarrow$ (2): By definition, a triangulated manifold is PL if and only if all vertex links are either PL balls or PL spheres. By Theorem 5.2, this is equivalent to (2).
(2) $\Rightarrow$ (3): Label all vertices of M by $v_1, \ldots, v_n$. Denote by L_i the link of v_i in M. By (1), there are integers $m_1, \ldots, m_n$ such that $\mathrm{sd}^{\,m_i} L_i$ is shellable. Since barycentric subdivisions preserve shellability, if we set $m = \max(m_1, \ldots, m_n)$, then $\mathrm{sd}^{\,m} L_i$ is shellable for all i.
(3) $\Rightarrow$ (4): Choose an m for which (3) holds. All of the vertex links in $\mathrm{sd}^{\,m} M$ are of the form $\mathrm{susp}^k \, \mathrm{sd}^{\,m} \mathrm{Link}\,(\tau, M)$, where τ is some face of M. Since $\mathrm{sd}^{\,m} \mathrm{Link}\,(\tau, M)$ is shellable, and shellability is preserved by suspensions, $\mathrm{susp}^k \, \mathrm{sd}^{\,m} \mathrm{Link}\,(\tau, M)$ is shellable. $\square$

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